

# MOMENTS OF 2D PARABOLIC ANDERSON MODEL

YU GU, WEIJUN XU

ABSTRACT. In this note, we use the Feynman-Kac formula to derive a moment representation for the 2D parabolic Anderson model in small time, which is related to the intersection local time of planar Brownian motions.

KEYWORDS: Feynman-Kac formula, renormalization, intersection local time.

## 1. INTRODUCTION

The aim of this note is to study the existence of moments of the solution to the parabolic Anderson model (PAM) in two spatial dimensions, formally given by

$$(1.1) \quad \partial_t u = \frac{1}{2} \Delta u + u \cdot \xi, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2,$$

where  $\xi$  is the two dimensional spatial white noise, that is, a centered Gaussian process with covariance  $\mathbb{E}[\xi(x)\xi(y)] = \delta(x - y)$ .

The equation is well-posed in dimension 1, but the product between  $u$  and  $\xi$  becomes ill-defined as soon as  $d \geq 2$ . For  $d = 2$ , the solution  $u$  is defined in [6, 7, 9] as the limit of a sequence of the regularized and renormalized equations. More precisely, fix a symmetric mollifier  $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}^+$  with  $\rho(x) = \rho(-x)$  and  $\int \rho = 1$ . Let

$$\rho_\varepsilon(x) = \varepsilon^{-2} \rho(x/\varepsilon), \quad \xi_\varepsilon = \xi \star \rho_\varepsilon,$$

and consider the equation

$$(1.2) \quad \partial_t u_\varepsilon = \frac{1}{2} \Delta u_\varepsilon + (\xi_\varepsilon - C_\varepsilon) u_\varepsilon,$$

for some large constant  $C_\varepsilon$ . Then, for

$$(1.3) \quad C_\varepsilon = \frac{1}{\pi} |\log \varepsilon| + O(1),$$

the sequence of solutions  $\{u_\varepsilon\}$  converges to some limit  $u$  (independent of the mollification) in probability, and we call this limit  $u$  the solution to 2D PAM. In  $d = 3$ , the mollifier  $\rho_\varepsilon(x) = \varepsilon^{-3} \rho(x/\varepsilon)$ , and the renormalization constant takes the form  $C_\varepsilon = c_1 \varepsilon^{-1} + c_2 |\log \varepsilon| + O(1)$  [8]. For  $d \geq 4$ , one does not expect to make sense of (1.1).

So far, most of the results mentioned above focused on the existence of the solution and the convergence of the regularized PDE after renormalization. The statistical properties of  $u$  remains a challenge; see [1, 2, 4] for some relevant discussions. The goal of this note is to show that the  $n$ -th moment of the solution

$u$  to 2D PAM exists for small time, and we present a Feynman-Kac formula for  $\mathbb{E}[u^n]$  in the time interval it exists.

**1.1. Main results.** We first give a heuristic derivation of  $\mathbb{E}[u(t, x)]^n$  by writing down a representation for  $\mathbb{E}[u_\varepsilon(t, x)]^n$  and passing to the limit formally. Suppose  $u_\varepsilon(0, x) = u_0(x)$  for some continuous function  $u_0$  with  $\|u_0\|_\infty \leq 1$ , we write the solution to (1.2) by the Feynman-Kac formula

$$(1.4) \quad u_\varepsilon(t, x) = \mathbb{E}_{\mathbf{B}} \left[ u_0(x + B_t) \exp \left( \int_0^t \xi_\varepsilon(x + B_s) ds - C_\varepsilon t \right) \right].$$

Here,  $B_t$  is a standard planar Brownian motion starting from the origin and independent of the white noise  $\xi$ , and  $C_\varepsilon$  is the constant defined in (1.3). We use  $\mathbb{E}_{\mathbf{B}}$  to denote the expectation with respect to  $B$ . We now proceed to calculating the  $n$ -th moment of  $u_\varepsilon(t, x)$ . First of all, the covariance function of  $\xi_\varepsilon$  satisfies

$$\mathbb{E}[\xi_\varepsilon(x)\xi_\varepsilon(y)] = R_\varepsilon(x - y) := \varepsilon^{-2}R\left(\frac{x - y}{\varepsilon}\right),$$

where  $R = \rho \star \rho$ , and  $\rho$  is the mollifier used to regularize the noise  $\xi$ . Next, one raises the expression (1.4) to the  $n$ -th power, and take a further expectation with respect to  $\xi_\varepsilon$ . Since  $B$  is independent of  $\xi_\varepsilon$ , one can interchange this expectation with the one with respect to the Brownian motions, and get

$$(1.5) \quad \mathbb{E}[u_\varepsilon(t, x)^n] = \mathbb{E}_{\mathbf{B}} \left[ \exp(I_n^\varepsilon(t) - nC_\varepsilon t) \prod_{k=1}^n u_0(x + B_t^k) \right].$$

Here,  $\{B^k\}_{k=1, \dots, n}$  are independent Brownian motions, and  $\mathbb{E}_{\mathbf{B}}$  denotes the expectation with respect to these  $B^k$ 's. Also,  $I_n^\varepsilon(t)$  is given by

$$(1.6) \quad I_n^\varepsilon(t) = \sum_{k=1}^n \int_0^t \int_0^s R_\varepsilon(B_s^k - B_u^k) du ds + \sum_{1 \leq i < j \leq n} \int_0^t \int_0^s R_\varepsilon(B_s^i - B_u^j) ds du,$$

where  $R_\varepsilon(x) = \varepsilon^{-2}R(x/\varepsilon)$  converges to the Delta function as  $\varepsilon \rightarrow 0$ . It is well known (see for example [3, Chapter 2]) that each term in the second term above (when  $i \neq j$ ) converges to the mutual intersection local time of Brownian motion, formally written as  $\int_{[0, t]^2} \delta(B_s^i - B_u^j) ds du$ . The first term above (when one has the same Brownian motion in the argument of  $R_\varepsilon$ ) unfortunately does not converge as  $\varepsilon \rightarrow 0$ , but it does when one subtracts its mean (see [13, 14, 11]). Thus, we define

$$(1.7) \quad \nu_\varepsilon(t) = \int_0^t \int_0^s \mathbb{E}_{\mathbf{B}}[R_\varepsilon(B_s - B_u)] du ds,$$

and for every  $t \geq 0$ , we have

$$(1.8) \quad I_n^\varepsilon(t) - n\nu_\varepsilon(t) \rightarrow \mathcal{X}_n(t)$$

in probability, where  $\mathcal{X}_n(t)$  is a linear combination of self- and mutual-intersection local times of planar Brownian motions, formally written as

$$(1.9) \quad \begin{aligned} \mathcal{X}_n(t) = & \sum_{k=1}^n \int_0^t \int_0^s \left( \delta(B_s^k - B_u^k) - \mathbb{E}_{\mathbf{B}}[\delta(B_s^k - B_u^k)] \right) dud s \\ & + \sum_{1 \leq i < j \leq n} \int_0^t \int_0^s \delta(B_s^i - B_u^j) ds du. \end{aligned}$$

Note that we do not have the factor  $\frac{1}{2}$  in front of the first term since the integration is on the simplex rather than the square  $[0, t]^2$ . It is well known from [11] that  $\mathcal{X}_n(t)$  has exponential moments for small enough  $t$  (depending on  $n$ ). In order for the expression (1.5) to converge, one needs the divergent constant  $C_\varepsilon t$  coincides with  $\nu_\varepsilon(t)$ . A simple calculations shows that this is indeed the case up to an  $O(1)$  correction.

**Lemma 1.1.** *Give a choice of the constant  $C_\varepsilon$  in (1.3) (that is, fix the choice of the  $O(1)$  part), there exists constants  $\mu_1$  and  $\mu_2$  such that  $\nu_\varepsilon(t) - C_\varepsilon t \rightarrow t(\mu_1 + \mu_2 \log t)$  as  $\varepsilon \rightarrow 0$ .*

By (1.8) and Lemma 1.1, we have

$$\begin{aligned} I_n^\varepsilon(t) - nC_\varepsilon t &= I_n^\varepsilon(t) - n\nu_\varepsilon(t) + n(\nu_\varepsilon(t) - C_\varepsilon t) \\ &\rightarrow \mathcal{X}_n(t) + nt(\mu_1 + \mu_2 \log t) \end{aligned}$$

in probability. If the sequences  $\{u_\varepsilon(t, x)^n\}$  and  $\{e^{I_n^\varepsilon(t) - nC_\varepsilon t}\}$  are uniformly integrable, then we can pass both sides of (1.5) to the limit, and obtain

$$(1.10) \quad \mathbb{E}[u(t, x)^n] = \mathbb{E}_{\mathbf{B}} \left[ \exp(\mathcal{X}_n(t) + nt(\mu_1 + \mu_2 \log t)) \prod_{k=1}^n u_0(x + B_t^k) \right].$$

The rest of the note is to show the uniform integrability of  $\{u_\varepsilon(t, x)^n\}$  and  $\{e^{I_n^\varepsilon(t) - nC_\varepsilon t}\}$  for small time  $t$ , so (1.10) does hold. The precise statement is the following.

**Theorem 1.2.** *There exists a universal constant  $\delta$  such that for every  $n \geq 1$ , the  $n$ -th moment of  $u$  exists for  $t \in (0, \frac{\delta}{n^2})$ , and (1.10) holds in that time interval.*

*Remark 1.3.* For  $n = 1$ , the moment formula reads

$$\mathbb{E}[u(t, x)] = \mathbb{E}_{\mathbf{B}}[u_0(x + B_t) e^{\gamma([0, t]_{<}^2) + t(\mu_1 + \mu_2 \log t)}],$$

with  $\gamma([0, t]_{<}^2) = \int_0^t \int_0^s (\delta(B_s - B_u) - \mathbb{E}_{\mathbf{B}}[\delta(B_s - B_u)]) dud s$  representing the self-intersection local time of  $B$ . It was proved in [12] that there exists  $t_0 > 0$  such that

$$\mathbb{E}_{\mathbf{B}}[e^{\gamma([0, t]_{<}^2)}] \begin{cases} < \infty & t < t_0, \\ = \infty & t > t_0. \end{cases}$$

Thus, it is natural to expect that the moments of  $u$  does not exist for large  $t$ , although we do not have a rigorous proof for it.

*Remark 1.4.* In [1], the authors defined the 2D Anderson Hamiltonian  $\mathcal{H} = -\Delta + \xi$  on the torus  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  using para-controlled calculus. An interesting

application is the exponential tail bounds for the ground state eigenvalue  $\Lambda_1$ . It was proved in [1, Proposition 5.4] that there exists  $C_1, C_2 > 0$  such that

$$e^{C_1 x} \leq \mathbb{P}[\Lambda_1 \leq x] \leq e^{C_2 x}$$

as  $x \rightarrow -\infty$ . Using the orthonormal eigenvectors of  $\mathcal{H}$ , denoted by  $\{e_n\}$ , we write the solution to PAM as

$$u(t, x) = \sum_{n=1}^{\infty} e^{-\Lambda_n t} \langle u_0, e_n \rangle e_n(x),$$

therefore,

$$\int_{\mathbb{T}^2} \mathbb{E}[|u(t, x)|^2] dx \leq \mathbb{E}[e^{-2\Lambda_1 t}] \int_{\mathbb{T}^2} |u_0(t, x)|^2 dx.$$

By the exponential tail bounds on  $\Lambda_1$ , it is clear the r.h.s. of the above display is only finite for small  $t$ , which is consistent with our result.

*Remark 1.5.* In the forthcoming article [5], the authors consider the 2D PAM with a small noise

$$(1.11) \quad \partial_t u = \Delta u + \beta u \cdot \xi, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2.$$

They obtain an explicit chaos expansion of certain polymer measure associated with (1.11) for  $\beta \ll 1$ . In particular, this implies that the second moment of  $u$  exists for  $t \in [0, 1], x \in \mathbb{R}^2$  and  $\beta$  sufficiently small. The restriction of  $\beta \ll 1$  is equivalent with our small time restriction. Indeed, define

$$u_\beta(t, x) := u(t/\beta^2, x/\beta),$$

one sees that  $u_\beta$  satisfies (1.1), hence for  $u_\beta(t, x)$  to be square integrable, we need  $t/\beta^2 \leq 1$ , i.e.,  $t \leq \beta^2 \ll 1$ .

*Remark 1.6.* A simple calculation shows that the moments of the approximations to 3D PAM explode as  $\varepsilon \rightarrow 0$ , and indicates that the solution to 3D PAM may not have a moment. To see this, we consider the constant initial condition  $u_0 \equiv 1$ , so

$$\mathbb{E}[u_\varepsilon(t, x)] = e^{-C_\varepsilon t} \mathbb{E}_{\mathbf{B}} \left[ \exp \left( \int_0^t \int_0^s R_\varepsilon(B_s - B_u) du ds \right) \right],$$

where  $R_\varepsilon(x) = \varepsilon^{-3} R(x/\varepsilon)$ . Since  $R(x)$  is continuous and  $R(0) > 0$ , we assume for simplicity that  $R(x) > \delta > 0$  for  $|x| \leq 2$ . Thus, by considering the event that  $|B_s| < \varepsilon$  for all  $s \in [0, t]$ , we have

$$\mathbb{E}_{\mathbf{B}} \left[ \exp \left( \int_0^t \int_0^s R_\varepsilon(B_s - B_u) du ds \right) \right] \geq \exp \left( \frac{\delta t^2}{2\varepsilon^3} \right) \mathbb{P} \left[ \sup_{s \in [0, t]} |B_s| < \varepsilon \right].$$

The probability  $\mathbb{P}[\sup_{s \in [0, t]} |B_s| < \varepsilon] \sim e^{-c' t \varepsilon^{-2}}$  for some  $c' > 0$  depending on the dimension. When  $d = 3$ , the renormalization constant  $C_\varepsilon = c_1 \varepsilon^{-1} + c_2 |\log \varepsilon| + O(1)$ . It implies that for any  $t > 0, x \in \mathbb{R}^3$ , we have  $\lim_{\varepsilon \rightarrow 0} \mathbb{E}[u_\varepsilon(t, x)] = \infty$ . The same discussion applies to  $d = 2$ , where

$$\mathbb{E}[u_\varepsilon(t, x)] \geq \exp \left( \frac{\delta t^2}{2\varepsilon^2} - \frac{c' t}{\varepsilon^2} - C_\varepsilon t \right).$$

If  $t > 2c'/\delta$ , we also have  $\lim_{\varepsilon \rightarrow 0} \mathbb{E}[u_\varepsilon(t, x)] = \infty$ .

*Remark 1.7.* When  $d = 2$ , the small time constraint for the existence of moments in our context also appears in [10, Theorem 4.1], where the usual product  $u \cdot \xi$  is replaced by the Wick product  $u \diamond \xi$ .

## 2. PROOF OF LEMMA 1.1 AND THEOREM 1.2

We denote  $[0, t]_{<}^n = \{0 \leq s_1 < \dots < s_n \leq t\}$ , and write  $a \lesssim b$  if  $a \leq Cb$  with some constant  $C$  independent of  $\varepsilon$ .

*Proof of Lemma 1.1.* By scaling property of Brownian motion, we have

$$R_\varepsilon(B_s - B_u) = \varepsilon^{-2} R\left(\frac{B_s - B_u}{\varepsilon}\right) \stackrel{\text{law}}{=} \varepsilon^{-2} R(B_{s/\varepsilon^2} - B_{u/\varepsilon^2}).$$

A change of variable  $(u/\varepsilon^2, s/\varepsilon^2) \mapsto (u, s)$  then yields

$$\nu_\varepsilon(t) = \varepsilon^2 \int_0^{t/\varepsilon^2} \int_0^s \mathbb{E}_{\mathbf{B}}[R(B_s - B_u)] du ds.$$

Now,  $B_s - B_u$  has the standard normal density  $(2\pi(s-u))^{-1} e^{-\frac{|x|^2}{2(s-u)}}$ . We then do another change of variable  $s-u \mapsto v$ , integrate  $s$  out, and rescale  $v \rightarrow v\varepsilon^2$ . This leads us to

$$\begin{aligned} \nu_\varepsilon(t) &= \frac{t}{2\pi} \int_{\mathbb{R}^2} R(x) \left( \int_0^t v^{-1} e^{-\frac{\varepsilon^2 |x|^2}{2v}} dv \right) dx - \frac{1}{2\pi} \int_0^t \left( \int_{\mathbb{R}^2} R(x) e^{-\frac{\varepsilon^2 |x|^2}{2v}} dx \right) dv \\ &:= \text{(i)} - \text{(ii)}. \end{aligned}$$

Since  $R$  integrates to 1, it is clear that

$$\text{(ii)} \rightarrow \frac{t}{2\pi}$$

as  $\varepsilon \rightarrow 0$ . As for (i), a substitution of variable  $\frac{\varepsilon^2 |x|^2}{2v} \mapsto \lambda$  and then an integration by parts yields

$$\begin{aligned} \text{(i)} &= \frac{t}{2\pi} \int_{\mathbb{R}^2} R(x) \left( \int_{\frac{\varepsilon^2 |x|^2}{2t}}^\infty \lambda^{-1} e^{-\lambda} d\lambda \right) dx \\ &= \frac{t}{2\pi} \int_{\mathbb{R}^2} R(x) \left( \int_{\frac{\varepsilon^2 |x|^2}{2t}}^\infty e^{-\lambda} \log \lambda d\lambda - e^{-\frac{\varepsilon^2 |x|^2}{2t}} \log \left( \frac{\varepsilon^2 |x|^2}{2t} \right) \right) dx. \end{aligned}$$

It is clear from the above expression that as  $\varepsilon \rightarrow 0$ , the only divergent part of (i) is from the term  $\log(\varepsilon^2)$ , and a direct calculation shows

$$\nu_\varepsilon(t) - \frac{t}{\pi} \cdot |\log \varepsilon| \rightarrow \mu_1 t + \mu_2 t \log t$$

for some constant  $\mu_1, \mu_2$ .  $\square$

*Proof of Theorem 1.2.* Fix  $(t, x)$  and  $n$ , and recall that

$$(2.1) \quad \mathbb{E}[u_\varepsilon(t, x)^n] = \mathbb{E}_{\mathbf{B}} \left[ \exp(I_n^\varepsilon(t) - nC_\varepsilon t) \prod_{k=1}^n u_0(x + B_t^k) \right],$$

where  $\mathbb{E}_{\mathbf{B}}$  is the expectation with respect to independent planar Brownian motions  $B^k$ 's, and  $I_n^\varepsilon$  is given by the expression (1.6). Note that  $u_\varepsilon(t, x)^n \rightarrow u(t, x)^n$  in probability, and that by (1.8) and Lemma 1.1, we have

$$I_n^\varepsilon(t) - nC_\varepsilon t \rightarrow \mathcal{X}_n(t) + nt(\mu_1 + \mu_2 \log t)$$

in probability. Thus, in view of (2.1), it suffices to show the uniform integrability of  $u_\varepsilon(t, x)^n$  and  $\exp(I_n^\varepsilon(t) - nC_\varepsilon t) \prod_{k=1}^n u_0(x + B_t^k)$ . This allows us to pass both sides of (2.1) to the limit and conclude Theorem 1.2.

To prove the uniform integrability, we bound the second moment of these two objects:

$$\mathbb{E}[|u_\varepsilon(t, x)|^{2n}] \lesssim \mathbb{E}_{\mathbf{B}}[e^{I_{2n}^\varepsilon(t) - 2nC_\varepsilon t}] \lesssim \mathbb{E}_{\mathbf{B}}[e^{I_{2n}^\varepsilon(t) - 2n\nu_\varepsilon(t)}],$$

and

$$\mathbb{E}_{\mathbf{B}}\left[\left|e^{I_n^\varepsilon(t)} e^{-nC_\varepsilon t} \prod_{k=1}^n u_0(x + B_t^k)\right|^2\right] \lesssim \mathbb{E}_{\mathbf{B}}[e^{2I_n^\varepsilon(t) - 2nC_\varepsilon t}] \lesssim \mathbb{E}_{\mathbf{B}}[e^{2(I_n^\varepsilon(t) - n\nu_\varepsilon(t))}],$$

where we have used  $\|u_0\|_\infty \leq 1$ . Thus, it suffices to show that for every  $n$  and  $\theta$ , there exists  $t_0$  small enough such that  $\mathbb{E}_{\mathbf{B}}[e^{\theta(I_n^\varepsilon(t) - n\nu_\varepsilon(t))}]$  is uniformly bounded in  $\varepsilon$  for all  $t < t_0$ . To see this, using Hölder's inequality, we get

$$\mathbb{E}_{\mathbf{B}}[e^{\theta(I_n^\varepsilon(t) - n\nu_\varepsilon(t))}] \leq \prod_{k=1}^n \left[ \mathbb{E}_{\mathbf{B}} e^{\theta N[\beta_\varepsilon^k([0, t]_\<^2) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon^k([0, t]_\<^2)]} \right]^{\frac{1}{N}} \prod_{1 \leq i < j \leq n} \left( \mathbb{E}_{\mathbf{B}} e^{\theta N \alpha_\varepsilon^{i,j}([0, t]^2)} \right)^{\frac{1}{N}},$$

where  $N = \frac{n(n+1)}{2}$ , and we have used the notations

$$\beta_\varepsilon^k([0, t]_\<^2) = \int_0^t \int_0^s R_\varepsilon(B_s^k - B_u^k) du ds, \quad \alpha_\varepsilon^{i,j}([0, t]^2) = \int_0^t \int_0^s R_\varepsilon(B_s^i - B_u^j) ds du.$$

By change of variables and the scaling property of the Brownian motion, we have

$$\beta_\varepsilon^k([0, t]_\<^2) \stackrel{\text{law}}{=} t \beta_{\varepsilon/\sqrt{t}}^k([0, 1]_\<^2), \quad \alpha_\varepsilon^{i,j}([0, t]^2) \stackrel{\text{law}}{=} t \alpha_{\varepsilon/\sqrt{t}}^{i,j}([0, 1]^2).$$

Then, Lemma A.1 implies that there exists  $\lambda, C > 0$  such that

$$t < \frac{\lambda}{\theta N} \Rightarrow \sup_{\varepsilon > 0} \mathbb{E}_{\mathbf{B}}[e^{\theta(I_n^\varepsilon(t) - n\nu_\varepsilon(t))}] \leq C.$$

This completes the proof.  $\square$

## APPENDIX A. EXPONENTIAL MOMENTS OF INTERSECTION LOCAL TIME OF PLANAR BROWNIAN MOTIONS

Recall that  $R_\varepsilon(x) = \varepsilon^{-2} R(\frac{x}{\varepsilon})$ , we define

$$\alpha_\varepsilon(A) = \int_A R_\varepsilon(B_s^1 - B_u^2) ds du, \quad \beta_\varepsilon(A) = \int_A R_\varepsilon(B_s - B_u) ds du$$

for any set  $A \subset \mathbb{R}_+^2$ , and

$$X_\varepsilon = \beta_\varepsilon([0, 1]_\<^2) - \mathbb{E}_{\mathbf{B}}[\beta_\varepsilon([0, 1]_\<^2)], \quad Y_\varepsilon = \alpha_\varepsilon([0, 1]^2).$$

**Lemma A.1.** *There exists universal constants  $\lambda, C > 0$  such that*

$$\sup_{\varepsilon > 0} (\mathbb{E}_{\mathbf{B}}[e^{\lambda X_\varepsilon}] + \mathbb{E}_{\mathbf{B}}[e^{\lambda Y_\varepsilon}]) \leq C.$$

The above result is standard. The case  $\varepsilon = 0$ , i.e., the exponential integrability of intersection local time, was addressed in the classical work [12]. We could not find a direct reference for  $\varepsilon > 0$ , though the proof follows essentially in the same line as the case of  $\varepsilon = 0$ . For the convenience of the reader, we present the details here.

*Proof.* We consider  $Y_\varepsilon$  first. Since  $R = \rho \star \rho$ , we can write

$$Y_\varepsilon = \int_{[0,1]^2} \int_{\mathbb{R}^2} \rho_\varepsilon(B_s^1 - x) \rho_\varepsilon(B_u^2 - x) dx ds du,$$

with  $\rho_\varepsilon(x) = \varepsilon^{-2} \rho(x/\varepsilon)$ . For any  $n \in \mathbb{N}$ ,

$$\begin{aligned} \mathbb{E}_{\mathbf{B}}[Y_\varepsilon^n] &= \int_{\mathbb{R}^{2n}} \left( \int_{[0,1]^{2n}} \mathbb{E}_{\mathbf{B}} \left[ \prod_{k=1}^n \rho_\varepsilon(B_{s_k}^1 - x_k) \rho_\varepsilon(B_{u_k}^2 - x_k) \right] ds du \right) d\mathbf{x} \\ &= \int_{\mathbb{R}^{2n}} \left( \int_{[0,1]^n} \mathbb{E}_{\mathbf{B}} \left[ \prod_{k=1}^n \rho_\varepsilon(B_{s_k} - x_k) \right] ds \right)^2 d\mathbf{x}. \end{aligned}$$

By [3, (2.2.11)], we have

$$\mathbb{E}_{\mathbf{B}}[Y_\varepsilon^n] = \int_{\mathbb{R}^{2n}} \left( \int_{\mathbb{R}^{2n}} \prod_{k=1}^n \rho_\varepsilon(z_k - x_k) \sum_{\sigma} \int_{[0,1]^n} \prod_{k=1}^n p_{s_k - s_{k-1}}(z_{\sigma(k)} - z_{\sigma(k-1)}) ds dz \right)^2 d\mathbf{x},$$

where  $p_t(x)$  is the density of  $N(0, t)$ ,  $[0, t]_{<}^n = \{0 \leq s_1 < \dots < s_n \leq t\}$ , and  $\sum_{\sigma}$  denotes the summation over all permutations over  $\{1, \dots, n\}$ . If we denote

$$h(z_1, \dots, z_n) = \sum_{\sigma} \int_{[0,1]^n} \prod_{k=1}^n p_{s_k - s_{k-1}}(z_{\sigma(k)} - z_{\sigma(k-1)}) ds, \quad Q_\varepsilon(z_1, \dots, z_n) = \prod_{k=1}^n \rho_\varepsilon(z_k),$$

then  $\mathbb{E}_{\mathbf{B}}[Y_\varepsilon^n]$  equals to

(A.1)

$$\begin{aligned} \int_{\mathbb{R}^{2n}} |Q_\varepsilon \star h(x_1, \dots, x_n)|^2 d\mathbf{x} &\leq \left( \int_{\mathbb{R}^{2n}} Q_\varepsilon(x_1, \dots, x_n) d\mathbf{x} \right)^2 \int_{\mathbb{R}^{2n}} |h(x_1, \dots, x_n)|^2 d\mathbf{x} \\ &= \int_{\mathbb{R}^{2n}} |h(x_1, \dots, x_n)|^2 d\mathbf{x} = \mathbb{E}_{\mathbf{B}}[\alpha([0, 1]^2)^n], \end{aligned}$$

where  $\alpha([0, 1]^2)$  is the mutual-intersection local time formally written as

$$\alpha([0, 1]^2) = \int_0^1 \int_0^1 \delta(B_s^1 - B_u^2) ds du,$$

and we used the Le Gall's moment formula in the second line of (A.1). By [12], we have

$$\mathbb{E}_{\mathbf{B}}[\exp(\mu \alpha([0, 1]^2))] < C$$

for some  $\mu > 0$ , hence we only need to choose  $\lambda = \mu$  to get

$$\mathbb{E}_{\mathbf{B}}[e^{\lambda Y_\varepsilon}] = \sum_{n=0}^{\infty} \frac{\lambda^n \mathbb{E}_{\mathbf{B}}[Y_\varepsilon^n]}{n!} \leq \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbf{B}}[|\alpha([0, 1]^2)|^n] = \mathbb{E}_{\mathbf{B}}[e^{\mu \alpha([0, 1]^2)}] < \infty.$$

Next, we consider  $X_\varepsilon$ . We define the triangle approximation of  $\{(u, s) : 0 \leq u < s \leq 1\}$ :

$$A_l^k = \left[ \frac{2l}{2^{k+1}}, \frac{2l+1}{2^{k+1}} \right) \times \left[ \frac{2l+1}{2^{k+1}}, \frac{2l+2}{2^{k+1}} \right), \quad l = 0, 1, \dots, 2^{k-1}, k = 0, 1, \dots$$

We will use the following three properties:

- (i) Fix any  $k$ ,  $\{\beta_\varepsilon(A_l^k)\}_{l=0, \dots, 2^{k-1}}$  are i.i.d. random variables.
- (ii)  $\beta_\varepsilon(A_l^k) \stackrel{\text{law}}{=} 2^{-(k+1)} \beta_{\varepsilon 2^{(k+1)/2}}([0, 1] \times [1, 2]) \stackrel{\text{law}}{=} 2^{-(k+1)} \alpha_{\varepsilon 2^{(k+1)/2}}([0, 1]^2)$
- (iii)  $\sup_{\varepsilon > 0} \mathbb{E}_{\mathbf{B}}[e^{\lambda \alpha_\varepsilon([0, 1]^2)}] \leq C$  for some  $\lambda, C > 0$ .

By (iii) and a Taylor expansion, there exists  $C > 0$  such that for sufficiently small  $\lambda$

$$(A.2) \quad \sup_{\varepsilon > 0} \mathbb{E}_{\mathbf{B}}[e^{\lambda(\alpha_\varepsilon([0, 1]^2) - \mathbb{E}_{\mathbf{B}}[\alpha_\varepsilon([0, 1]^2)])}] \leq e^{C\lambda^2}.$$

We fix the constants  $\lambda, C$  from now on, and write

$$X_\varepsilon = \sum_{k=0}^{\infty} \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}}[\beta_\varepsilon(A_l^k)]).$$

Fix  $a \in (0, 1)$  and define a sequence of constants

$$b_1 = 2\lambda, \quad b_N = 2\lambda \prod_{j=2}^N (1 - 2^{-a(j-1)}), \quad N = 2, 3, \dots,$$

we have

$$\begin{aligned} & \mathbb{E}_{\mathbf{B}} \exp \left[ b_N \sum_{k=0}^N \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_l^k)) \right] \\ & \leq \left( \mathbb{E}_{\mathbf{B}} \exp \left[ b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_l^k)) \right] \right)^{1-2^{-a(N-1)}} \\ & \quad \times \left( \mathbb{E}_{\mathbf{B}} \exp \left[ 2^{a(N-1)} b_N \sum_{l=0}^{2^N-1} (\beta_\varepsilon(A_l^N) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_l^N)) \right] \right)^{2^{-a(N-1)}} \\ & \leq \mathbb{E}_{\mathbf{B}} \exp \left[ b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_l^k)) \right] \\ & \quad \times \left( \mathbb{E}_{\mathbf{B}} \exp \left[ 2^{a(N-1)} b_N (\beta_\varepsilon(A_0^N) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_0^N)) \right] \right)^{2^{N-a(N-1)}}. \end{aligned}$$

Since  $\beta_\varepsilon(A_0^N) \stackrel{\text{law}}{=} 2^{-(N+1)} \alpha_{\varepsilon 2^{(N+1)/2}}([0, 1]^2)$ , we have

$$\begin{aligned} & \mathbb{E}_{\mathbf{B}} \exp \left[ 2^{a(N-1)} b_N (\beta_\varepsilon(A_0^N) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_0^N)) \right] \\ & = \mathbb{E}_{\mathbf{B}} \exp \left[ 2^{a(N-1)} b_N 2^{-(N+1)} (\alpha_{\varepsilon 2^{(N+1)/2}}([0, 1]^2) - \mathbb{E}_{\mathbf{B}} \alpha_{\varepsilon 2^{(N+1)/2}}([0, 1]^2)) \right]. \end{aligned}$$



Using the fact that  $2^{a(N-1)}b_N2^{-(N+1)} < \lambda$  and (A.2), we derive for all  $\varepsilon > 0$  that

$$\mathbb{E}_{\mathbf{B}} \exp \left[ 2^{a(N-1)}b_N(\beta_\varepsilon(A_0^N) - \mathbb{E}_{\mathbf{B}}\beta_\varepsilon(A_0^N)) \right] \leq e^{Cb_N^2 2^{-2N+2a(N-1)}},$$

so there exists  $C' > 0$  such that

$$\begin{aligned} & \mathbb{E}_{\mathbf{B}} \exp \left[ b_N \sum_{k=0}^N \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}}\beta_\varepsilon(A_l^k)) \right] \\ & \leq \mathbb{E}_{\mathbf{B}} \exp \left[ b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}}\beta_\varepsilon(A_l^k)) \right] e^{C'2^{(a-1)N}}. \end{aligned}$$

Iterating the above inequality, we get

$$\mathbb{E}_{\mathbf{B}} \exp \left[ b_N \sum_{k=0}^N \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}}\beta_\varepsilon(A_l^k)) \right] \leq \exp(C'(1 - 2^{a-1})^{-1})$$

Since  $b_N \rightarrow b_\infty$  for some  $b_\infty > 0$ , we have

$$\mathbb{E}_{\mathbf{B}}[\exp(b_\infty X_\varepsilon)] \leq \exp(C'(1 - 2^{a-1})^{-1}),$$

which completes the proof.  $\square$

**Acknowledgments.** We thank Dirk Erhard and Nikolaos Zygouras for stimulating discussions and for showing us the argument in Remark 1.6. YG is partially supported by the NSF through DMS-1613301. WX is supported by EPSRC through the research fellowship EP/N021568/1.

## REFERENCES

- [1] R. ALLEZ AND K. CHOUK, *The continuous Anderson hamiltonian in dimension two*, arXiv preprint arXiv:1511.02718, (2015).
- [2] G. CANNIZZARO AND K. CHOUK, *Multidimensional SDEs with singular drift and universal construction of the polymer measure with white noise potential*, arXiv preprint arXiv:1501.04751, (2015).
- [3] X. CHEN, *Random walk intersections: Large deviations and related topics*, no. 157, American Mathematical Soc., 2010.
- [4] K. CHOUK, J. GAIRING, AND N. PERKOWSKI, *An invariance principle for the two-dimensional parabolic Anderson model with small potential*, arXiv preprint arXiv:1609.02471, (2016).
- [5] D. ERHARD AND N. ZYGOURAS, *private communication*.
- [6] M. GUBINELLI, P. IMKELLER, AND N. PERKOWSKI, *Paracontrolled distributions and singular pdes*, in Forum of Mathematics, Pi, vol. 3, Cambridge Univ Press, 2015, p. e6.
- [7] M. HAIRER, *A theory of regularity structures*, Inventiones mathematicae, 198 (2014), pp. 269–504.
- [8] M. HAIRER AND C. LABBÉ, *Multiplicative stochastic heat equations on the whole space*, arXiv preprint arXiv:1504.07162, (2015).
- [9] ———, *A simple construction of the continuum parabolic anderson model on  $\mathbf{R}^2$* , Electronic Communications in Probability, 20 (2015).
- [10] Y. HU, *Chaos expansion of heat equations with white noise potentials*, Potential Analysis, 16 (2002), pp. 45–66.
- [11] J.-F. LE GALL, *Some properties of planar brownian motion*, in Ecole d’Eté de Probabilités de Saint-Flour XX-1990, Springer, 1992, pp. 111–229.

- [12] ———, *Exponential moments for the renormalized self-intersection local time of planar brownian motion*, in Séminaire de Probabilités XXVIII, Springer, 1994, pp. 172–180.
- [13] S. VARADHAN, *Appendix to euclidean quantum field theory by k. symanzik*, Local Quantum Theory. Academic Press, Reading, MA, 1 (1969).
- [14] M. YOR, *Precisions sur l'existence et la continuité des temps locaux d'intersection du mouvement brownien dans  $\mathbf{R}^2$* , in Séminaire de Probabilités XX 1984/85, Springer, 1986, pp. 532–542.

(Yu Gu) DEPARTMENT OF MATHEMATICS, BUILDING 380, STANFORD UNIVERSITY, STANFORD, CA, 94305, USA

(Weijun Xu) MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL, UK